

第9讲：三角函数公式化简+题型拓展+技巧衍生(2)

题型一：二倍角+半角公式+降幂公式+三倍角公式

$$\sin 2\alpha = \sin(\alpha + \alpha) = 2\sin\alpha \cos\alpha \Rightarrow \sin\alpha \cos\alpha = \frac{1}{2} \sin 2\alpha$$

$$\cos 2\alpha = \cos(\alpha + \alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 \Rightarrow \begin{cases} \sin^2\alpha = \frac{1 - \cos 2\alpha}{2} \\ \cos^2\alpha = \frac{1 + \cos 2\alpha}{2} \end{cases}$$

$$\tan 2\alpha = \tan(\alpha + \alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1 + 2\cos^2\frac{\alpha}{2} - 1} = \tan\frac{\alpha}{2}$$

总结：① 1和sin用 $\Rightarrow$ 凑完全平方 ② 1和cos用 $\Rightarrow$ 倍角公式降

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$$\begin{aligned} \text{三倍角: } \sin 3\alpha &= \sin(2\alpha + \alpha) = \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha = 2\sin\alpha \cdot (1 - \sin^2\alpha) + (1 - 2\sin^2\alpha) \cdot \sin\alpha \\ &= 3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha \cdot \left(\frac{3}{4} - \sin^2\alpha\right) = 4\sin\alpha \cdot \left(\frac{\sqrt{3}}{2} + \sin\alpha\right) \cdot \left(\frac{\sqrt{3}}{2} - \sin\alpha\right) \\ &= 4\sin\alpha \cdot (\sin 60^\circ + \sin\alpha) \cdot (\sin 60^\circ - \sin\alpha) = 4\sin\alpha \cdot 2\sin\left(30^\circ + \frac{\alpha}{2}\right) \cos\left(30^\circ - \frac{\alpha}{2}\right) \cdot 2\cos\left(30^\circ + \frac{\alpha}{2}\right) \sin\left(30^\circ - \frac{\alpha}{2}\right) \\ &= 4\sin\alpha \cdot \sin(60^\circ + \alpha) \cdot \sin(60^\circ - \alpha) = 4\sin(60^\circ - \alpha) \cdot \sin\alpha \cdot \sin(60^\circ + \alpha) \end{aligned}$$

3角半

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题型一：二倍角+半角公式+降幂公式+三倍角公式

$$\cos 3\alpha = \cos(2\alpha + \alpha) = \cos 2\alpha \cdot \cos\alpha - \sin 2\alpha \cdot \sin\alpha = (2\cos^2\alpha - 1) \cdot \cos\alpha - 2\cos\alpha \cdot (1 - \cos^2\alpha)$$

$$= -3\cos\alpha + 4\cos^3\alpha = 4\cos\alpha \cdot \left(\cos^2\alpha - \frac{3}{4}\right) = 4\cos\alpha \cdot \left(\cos\alpha + \frac{\sqrt{3}}{2}\right) \cdot \left(\cos\alpha - \frac{\sqrt{3}}{2}\right)$$

$$\begin{aligned} &= 4\cos\alpha \cdot (\cos\alpha + \cos 30^\circ) \cdot (\cos\alpha - \cos 30^\circ) = 4\cos\alpha \cdot 2\cos\left(\frac{\alpha}{2} + 15^\circ\right) \cdot 2\cos\left(\frac{\alpha}{2} - 15^\circ\right) \cdot (-2)\sin\left(\frac{\alpha}{2} + 15^\circ\right) \cdot \sin\left(\frac{\alpha}{2} - 15^\circ\right) \\ &= 4\cos\alpha \cdot \sin(\alpha + 30^\circ) \sin(30^\circ - \alpha) = 4\cos(60^\circ - \alpha) \cdot \cos\alpha \cdot \cos(60^\circ + \alpha) \end{aligned}$$

$$\therefore \tan 3\alpha = \frac{\sin 3\alpha}{\cos 3\alpha} = \frac{4\sin(60^\circ - \alpha) \cdot \sin\alpha \cdot \sin(60^\circ + \alpha)}{4\cos(60^\circ - \alpha) \cdot \cos\alpha \cdot \cos(60^\circ + \alpha)} = \tan(60^\circ - \alpha) \cdot \tan\alpha \cdot \tan(60^\circ + \alpha)$$

3角半

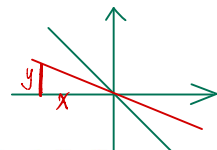
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1. (2024届九省联考) 已知 $\theta \in \left(\frac{3\pi}{4}, \pi\right)$, $\tan 2\theta = -4\tan\left(\theta + \frac{\pi}{4}\right)$, 则 $\frac{1 + \sin 2\theta}{2\cos^2\theta + \sin 2\theta} = (\quad)$

A. $\frac{1}{4}$ B. $\frac{3}{4}$ C. 1 D. $\frac{3}{2}$

$$\begin{aligned} &\frac{1 + \sin 2\theta}{2\cos^2\theta + \sin 2\theta} = \frac{1 + 2\sin\theta \cos\theta}{2\cos^2\theta + 2\sin\theta \cos\theta} = \frac{(\sin\theta + \cos\theta)^2}{2\cos\theta \cdot (\cos\theta + \sin\theta)} = \frac{\sin\theta + \cos\theta}{2\cos\theta} = \frac{1}{2} (\tan\theta + 1) \end{aligned}$$

$$\frac{2\tan\theta}{1 - \tan^2\theta} = -4 \cdot \frac{\tan\theta + 1}{1 - \tan\theta} \Rightarrow \tan\theta = -2 \cdot (1 + \tan\theta)^2 = -2\tan^2\theta - 4\tan\theta - 2$$



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$$2\tan^2\theta + 5\tan\theta + 2 = 0$$

$$\frac{2}{1} \times \frac{1}{2} \quad \tan\theta = -2 \text{ or } -\frac{1}{2}$$

舍

2. (2023·新高考II) 已知 α 为锐角, $\cos\alpha = \frac{1+\sqrt{5}}{4}$, 则 $\sin\frac{\alpha}{2} = (\quad)$

- A. $\frac{3-\sqrt{5}}{8}$ B. $\frac{-1+\sqrt{5}}{8}$ C. $\frac{3-\sqrt{5}}{4}$ D. $\frac{-1+\sqrt{5}}{4}$

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

$$\sin^2\frac{\alpha}{2} = \frac{1 - \frac{1}{4} - \frac{\sqrt{5}}{4}}{2} = \frac{3-\sqrt{5}}{8} = \frac{6-2\sqrt{5}}{16} = \frac{\sqrt{5}^2 - 2\cdot\sqrt{5}\cdot 1 + 1}{16}$$

$$\sin\frac{\alpha}{2} = \frac{\sqrt{5}-1}{4} = \frac{(\sqrt{5}-1)^2}{16}$$

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3. $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \underline{\hspace{2cm}}$.

$$\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha \Rightarrow \cos\alpha = \frac{\sin 2\alpha}{2\sin\alpha}$$

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sin 40^\circ}{\sin 20^\circ} \cdot \frac{1}{2} \cdot \frac{\sin 80^\circ}{\sin 40^\circ} \cdot \frac{1}{2} \cdot \frac{\sin 160^\circ}{\sin 80^\circ} = \frac{1}{16}$$

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4. $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \underline{\hspace{2cm}}$.

$$\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha$$

$$= \sin 6^\circ \cdot \cos 12^\circ \cdot \cos 24^\circ \cdot \cos 48^\circ \quad \leftarrow \Rightarrow \cos\alpha = \frac{1}{2} \cdot \frac{\sin 2\alpha}{\sin\alpha}$$

$$= \sin 6^\circ \cdot \frac{1}{2} \cdot \frac{\sin 24^\circ}{\sin 12^\circ} \cdot \frac{1}{2} \cdot \frac{\sin 48^\circ}{\sin 24^\circ} \cdot \frac{1}{2} \cdot \frac{\sin 96^\circ}{\sin 48^\circ}$$

$$= \frac{2 \sin 6^\circ \cdot \cos 6^\circ}{2 \cdot 8 \cdot \sin 12^\circ} = \frac{1}{16}$$

$$\text{巧法: } = \frac{2 \sin 12^\circ \cdot \cos 6^\circ \sin 6^\circ \cdot \cos 12^\circ \cdot \cos 24^\circ \cdot \cos 48^\circ}{2 \cdot 2 \cos 6^\circ} = \frac{1}{16}$$

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5. (2025·德阳模拟) 若关于 α 的方程 $\frac{\cos 2\alpha + 1 + m \sin 2\alpha}{m \cos 2\alpha + m - \sin 2\alpha} = \frac{\sin^3 \alpha}{\cos^3 \alpha}$ 在区间 $(0, \frac{\pi}{4})$ 上有且仅有一个实数解, 则实数 $m = \underline{-2\sqrt{2}}$.

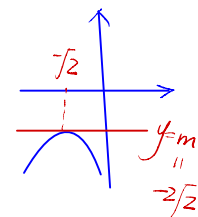
$$\frac{2\cos^2\alpha - 1 + 1 + 2m \cdot \sin\alpha \cos\alpha}{m \cdot (2\cos^2\alpha - 1 + 1) - 2\sin\alpha \cos\alpha} = \frac{\cos\alpha + m \sin\alpha}{m \cos\alpha - \sin\alpha} = \frac{1 + m \tan\alpha}{m - \tan\alpha} = \tan^3 \alpha$$

$$1 + m \cdot \tan\alpha = m \cdot \tan^3 \alpha - \tan^4 \alpha$$

$$m = \frac{\tan^4 \alpha + 1}{\tan^3 \alpha - \tan\alpha} = \frac{\tan^2 \alpha + \frac{1}{\tan^2 \alpha}}{\tan\alpha - \frac{1}{\tan\alpha}} = \frac{x^2 + 2}{x} = x + \frac{2}{x}$$

$$\text{令 } x = \tan\alpha - \frac{1}{\tan\alpha} = t - \frac{1}{t} \in (-\infty, 0)$$

$$x^2 = \tan^2 \alpha + \frac{1}{\tan^2 \alpha} - 2, \quad t = \tan\alpha \in (0, 1)$$



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6. (2025·莆田模拟) 已知 $\alpha \in (0, \pi)$, $2\sin\alpha + \cos\alpha = 1$, 则 $\frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}} = (\quad)$

A. 3

B. -3

C. $\frac{1}{3}$

D. $-\frac{1}{3}$

$$2\sin\alpha = 1 - \cos\alpha$$

$$4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 1 - (1 - 2\sin^2\frac{\alpha}{2}) = 2\sin^2\frac{\alpha}{2}$$

$$2\cos\frac{\alpha}{2} = \sin\frac{\alpha}{2} \Rightarrow \tan\frac{\alpha}{2} = 2$$

$$\text{公式: } \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

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7. (2025·湖北模拟) 若 $(1 + \cos\frac{2\pi}{5})\sin x = \sin\frac{2\pi}{5}\cos x$, 且 $x \in (0, \pi)$, 则 $x = (\quad)$

A. $\frac{\pi}{5}$

B. $\frac{3\pi}{10}$

C. $\frac{7\pi}{10}$

D. $\frac{4\pi}{5}$

$$\tan x = \frac{\sin\frac{2\pi}{5}}{1+\cos\frac{2\pi}{5}} = \frac{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}{1+2\cos^2\frac{\pi}{5}-1} = \tan\frac{\pi}{5}$$

$$\Rightarrow x = \frac{\pi}{5}$$

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8. (2024·江西八校联考) 已知 $\alpha, \beta \in (0, \frac{\pi}{2})$, $2(\sin\beta + \sin^2\beta) = \frac{\sin 2\beta}{\tan\alpha}$, 则 $\tan(2\alpha + \beta + \frac{\pi}{6}) = (\quad)$

A. $-\sqrt{3}$

B. $-\frac{\sqrt{3}}{3}$

C. $\frac{\sqrt{3}}{3}$

D. $\sqrt{3}$

$$2\sin\beta \cdot (1 + \sin\beta) = \frac{2\sin\beta \cdot \cos\beta}{\tan\alpha}$$

$$\tan\alpha = \frac{\cos\beta}{1+\sin\beta} = \frac{\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}}{1+2\sin\frac{\beta}{2}\cos\frac{\beta}{2}} = \frac{\cos\frac{\beta}{2} - \sin\frac{\beta}{2}}{\sin\frac{\beta}{2} + \cos\frac{\beta}{2}} = \frac{1 - \tan\frac{\beta}{2}}{1 + \tan\frac{\beta}{2}} = \frac{\tan\frac{\pi}{4} - \tan\frac{\beta}{2}}{1 + \tan\frac{\beta}{2} \cdot \tan\frac{\pi}{4}}$$

$$\alpha = \frac{\pi}{4} - \frac{\beta}{2} \Rightarrow 2\alpha + \beta = \frac{\pi}{2} \quad = \tan(\frac{\pi}{4} - \frac{\beta}{2})$$

$$\tan 120^\circ = -\sqrt{3}$$

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9. 化简 $\frac{\cos 40^\circ}{\cos 25^\circ \sqrt{1 - \sin 40^\circ}}$

$$\text{法一: } \frac{\cos 40^\circ}{\cos 25^\circ \cdot \sqrt{1 - \cos 50^\circ}} = \frac{\cos 40^\circ}{\sqrt{2} \sin 25^\circ \cdot \cos 25^\circ} = \frac{\sqrt{2} \cos 40^\circ}{\sin 50^\circ} = \sqrt{2}$$

$$\text{法二: } \frac{\cos 40^\circ}{\cos 25^\circ \cdot \sqrt{1 - 2\sin 20^\circ \cos 20^\circ}} = \frac{\cos 40^\circ}{\cos 25^\circ \cdot \sqrt{(\sin 20^\circ - \cos 20^\circ)^2}} = \frac{\cos 40^\circ}{\cos 25^\circ \cdot (\cos 20^\circ - \sin 20^\circ)} = \frac{\cos 20^\circ + \sin 20^\circ}{\cos 25^\circ} = \frac{\sqrt{2} \sin(20^\circ + 45^\circ)}{\cos 25^\circ} = \sqrt{2}$$

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10. 已知 $\sin \alpha = \frac{1}{2} + \cos \alpha$, 且 $\alpha \in (0, \frac{\pi}{2})$, 则 $\frac{\cos 2\alpha}{\sin(\alpha - \frac{\pi}{4})}$ 的值为 _____.

$$\begin{cases} \sin \alpha - \cos \alpha = \frac{1}{2} \\ \cos \alpha + \sin \alpha = t > 0 \end{cases}$$

$$1 + 1 + 0 = \frac{1}{4} + t^2$$

$$\Downarrow$$

$$t = \frac{\sqrt{7}}{2}$$

$$\frac{\cos^2 \alpha - \sin^2 \alpha}{\frac{\sqrt{2}}{2} (\sin \alpha - \cos \alpha)} = \frac{\cos \alpha + \sin \alpha}{-\frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{7}}{2}}{-\frac{\sqrt{2}}{2}} = -\frac{\sqrt{14}}{2}$$

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11. 已知 $\tan x = a$, 则 $\frac{3\sin x + \sin 3x}{3\cos x + \cos 3x} =$ _____.

$$\frac{3\sin x + 3\sin x - 4\sin^3 x}{3\cos x - 3\cos x + 4\cos^3 x} = \frac{6\sin x - 4\sin^3 x}{4\cos^3 x}$$

$$= \frac{6\sin x \cdot (\sin^2 x + \cos^2 x) - 4\sin^3 x}{4\cos^3 x} = \frac{6\tan x \cdot (\tan^2 x + 1) - 4\tan^3 x}{4} = \frac{6a \cdot (a^2 + 1) - 4a^3}{4}$$

$$= \frac{2a^3 + 6a}{4} = \frac{a^3 + 3a}{2}$$

赵礼显数学

题型二: 辅助角公式

$$y = a \sin wx + b \cos wx = \sqrt{a^2 + b^2} \cdot \left(\sin wx \cdot \frac{a}{\sqrt{a^2 + b^2}} + \cos wx \cdot \frac{b}{\sqrt{a^2 + b^2}} \right)$$

$$= \sqrt{a^2 + b^2} \sin(wx + \varphi) \quad \tan \varphi = \frac{b}{a} \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}$$

$$y = 3\sin 2x - \sqrt{3}\cos 2x = 2\sqrt{3} \sin(2x - \frac{\pi}{6}) \quad \tan \varphi = \frac{\sqrt{3}}{3}$$

$$y = \sin 2x + \sqrt{3}\cos 2x = 2\sin(2x + \frac{\pi}{3}) \quad \tan \varphi = \sqrt{3}$$

$$y = \cos 2x - \sqrt{3}\sin 2x = -(\sqrt{3}\sin 2x - \cos 2x) = -2\sin(2x - \frac{\pi}{6})$$

$$y = \cos 2x - \sqrt{3}\sin 2x = 2 \left(\cos 2x \cdot \frac{1}{2} - \sin 2x \cdot \frac{\sqrt{3}}{2} \right) = 2\cos(2x + \frac{\pi}{3})$$

$\tan \varphi = \frac{1}{\sqrt{3}}$

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12. (2020·新课标III) 已知 $\sin \theta + \sin(\theta + \frac{\pi}{3}) = 1$, 则 $\sin(\theta + \frac{\pi}{6}) =$ (B)

A. $\frac{1}{2}$

B. $\frac{\sqrt{3}}{3}$

C. $\frac{2}{3}$

D. $\frac{\sqrt{2}}{2}$

法一: 和差化积

$$\text{法二: } \sin \theta + \sin \theta \cdot \frac{1}{2} + \cos \theta \cdot \frac{\sqrt{3}}{2} = 1$$

$$\frac{3}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta = 1$$

$$\tan \theta = \frac{\sqrt{3}}{2}$$

$$\sqrt{3} \sin(\theta + \frac{\pi}{6}) = 1$$

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13. (2013·新课标 I) 设当 $x = \theta$ 时, 函数 $f(x) = \sin x - 2\cos x$ 取得最大值, 则 $\cos \theta =$ _____.

$$\max = \sqrt{5}$$

$$\sin \theta - 2\cos \theta = \sqrt{5} \Rightarrow 2\sin \theta - 4\cos \theta = 2\sqrt{5}$$

法一: 辅助角: $\cos \theta + 2\sin \theta = t = 0$

$$1 + 4 + 0 = 5 + t^2 \Rightarrow t = 0$$

$$\Rightarrow 5\cos \theta = -2\sqrt{5} \Rightarrow \cos \theta = -\frac{2\sqrt{5}}{5}$$

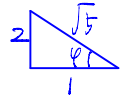
法二: $\sin \theta - 2\cos \theta = \sqrt{5} \Rightarrow \sin(\theta - \varphi) = 1$

$$\tan \varphi = 2$$

$$\sin(\theta - \varphi) = 1$$

$$\theta - \varphi = \frac{\pi}{2} + 2k\pi$$

$$\theta = \varphi + \frac{\pi}{2} + 2k\pi \Rightarrow \cos \theta = \cos(\varphi + \frac{\pi}{2}) = -\sin \varphi = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$



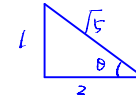
赵礼显数学

13. (2013·新课标 I) 设当 $x = \theta$ 时, 函数 $f(x) = \sin x - 2\cos x$ 取得最大值, 则 $\cos \theta =$ _____.

$$f'(x) = \cos x + 2\sin x$$

$$f'(\theta) = \cos \theta + 2\sin \theta = 0$$

$$\tan \theta = -\frac{1}{2} \Rightarrow \cos \theta = \pm \frac{2}{\sqrt{5}} = \pm \frac{2\sqrt{5}}{5}$$



$$\Rightarrow \begin{cases} \cos \theta = \frac{2\sqrt{5}}{5} \\ \sin \theta = -\frac{\sqrt{5}}{5} \end{cases} \text{ or } \begin{cases} \cos \theta = -\frac{2\sqrt{5}}{5} \\ \sin \theta = \frac{\sqrt{5}}{5} \end{cases}$$

min max

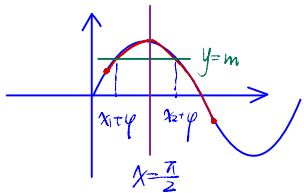
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14. 已知 x_1, x_2 是函数 $f(x) = 2\sin x + \cos x - m$ 在 $[0, \pi]$ 内的两个零点, 则 $\sin(x_1 + x_2) =$ _____.

$$2\sin x + \cos x = m$$

$$\sqrt{5} \sin(x + \varphi) = m \quad \tan \varphi = \frac{1}{2}$$

$$\sqrt{5} \sin u = m, \quad u \in [\varphi, \pi + \varphi]$$



$$x_1 + x_2 + 2\varphi = \pi \Rightarrow \sin(x_1 + x_2) = \sin(\pi - 2\varphi) = \sin 2\varphi = \frac{2\sin \varphi \cos \varphi}{\cos^2 \varphi + \sin^2 \varphi} = \frac{2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}}}{1 + \frac{1}{5}} = \frac{4}{5}$$

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15. (2025·湖北模拟) 已知 α, β 为锐角, 且 $\sin(\alpha + \beta) = \frac{\sin \alpha}{\sin \beta}$, 则 $\tan \alpha$ 的最大值为 _____.

$$\sin \alpha \cos \beta + \cos \alpha \sin \beta = \frac{\sin \alpha}{\sin \beta} \quad \text{同除 } \cos \alpha$$

$$\tan \alpha \cdot \cos \beta + \sin \beta = \tan \alpha \cdot \frac{1}{\sin \beta}$$

$$\tan \alpha \cdot \sin \beta \cos \beta + \sin^2 \beta = \tan \alpha$$

$$\tan \alpha = \frac{\sin^2 \beta}{1 - \sin \beta \cos \beta} = \frac{\sin^2 \beta}{\sin^2 \beta + \cos^2 \beta - \sin \beta \cos \beta} = \frac{\tan^2 \beta}{\tan^2 \beta + 1 - \tan \beta} = \frac{1}{1 + \frac{1}{\tan \beta} - \tan \beta}$$

$$\text{法二: } \frac{1 - \cos 2\beta}{2} = \frac{1 - \cos 2\beta}{2 - \sin 2\beta} = y$$

$$1 - \cos 2\beta = 2y - y \sin 2\beta \Rightarrow y \sin 2\beta - \cos 2\beta = 2y - 1$$

$$\sqrt{y^2 + 1} \sin(2\beta - \varphi) = 2y - 1, \quad \left| \frac{2y-1}{\sqrt{y^2+1}} \right| \leq 1 \Rightarrow 4y^2 - 4y + 1 \leq y^2 + 1 \Rightarrow 0 \leq y \leq \frac{4}{3}$$

$$\text{当 } \beta = 0 \text{ 或 } y = 0 \text{ 时, } \tan \alpha = 0. \quad \alpha, \beta \in \left(0, \frac{\pi}{2}\right)$$

$$x = \frac{1}{\tan \beta} > 0$$

$$\frac{1}{0} + \frac{1}{2} \leq x \leq \frac{1}{2} \quad \max: \frac{4}{3}$$

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15. (2025·湖北模拟) 已知 α, β 为锐角, 且 $\sin(\alpha+\beta) = \frac{\sin\alpha}{\sin\beta}$, 则 $\tan\alpha$ 的最大值为 _____.

$$\frac{1-\cos 2\beta}{2-\sin 2\beta} \Rightarrow k = \frac{2-\sin 2\beta}{1-\cos 2\beta}$$
 过 $(\cos 2\beta, \sin 2\beta), (1, 2)$ 两点, 在斜率. 单位圆上动点.

$$y-2=k(x-1)$$

$$d = \frac{|k-1|}{\sqrt{k^2+1}} = 1 \Rightarrow 4+k^2-4k=k^2+1 \Rightarrow k=\frac{3}{4}$$

$k \in [\frac{3}{4}, +\infty) \Rightarrow \sqrt{2} \sin(\alpha+\frac{\pi}{4}) > 1$

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16. 设 α, β 是一个钝角三角形的两个锐角, 下列四个不等式中正确的是 ()

- A. $\tan\alpha\tan\beta < 1$
 B. $\sin\alpha + \sin\beta < \sqrt{2}$
 C. $\cos\alpha + \cos\beta > 1$
 D. $\frac{1}{2}\tan(\alpha+\beta) < \tan\frac{\alpha+\beta}{2}$

A. $\sin\alpha\sin\beta < \cos\alpha\cos\beta \Leftrightarrow \cos(\alpha+\beta) > 0$

B. $\sin\alpha + \sin\beta \leq \sin\alpha + \sin(\frac{\pi}{2}-\alpha) = \sin\alpha + \cos\alpha = \sqrt{2}\sin(\alpha+\frac{\pi}{4}) \leq \sqrt{2}$

C. $\cos\alpha + \cos\beta > \cos\alpha + \cos(\frac{\pi}{2}-\alpha) = \sin\alpha + \cos\alpha = \sqrt{2}\sin(\alpha+\frac{\pi}{4}) > 1$

D. $\frac{1}{2} \cdot \frac{2\tan\frac{\alpha+\beta}{2}}{1-\tan^2\frac{\alpha+\beta}{2}} < \tan\frac{\alpha+\beta}{2} \Rightarrow 1-\tan^2\frac{\alpha+\beta}{2} > 1$

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题型三: 万能公式

$$\sin\alpha = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2t}{1+t^2} \quad t = \tan\frac{\alpha}{2}$$

$$\cos\alpha = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-t^2}{1+t^2}$$

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

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17. 已知 α 为第四象限角, $\sin\alpha + \cos\alpha = \frac{1}{5}$, 则 $\tan\frac{\alpha}{2}$ 的值为 ()

- A. $-\frac{1}{2}$
 B. $\frac{1}{2}$
 C. $-\frac{1}{3}$
 D. $\frac{1}{3}$

$$\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2t+1-t^2}{1+t^2} = \frac{1}{5}$$

$$10t+5-5t^2=1+t^2$$

$$6t^2-10t-4=0$$

$$3t^2-5t-2=0$$

$$t = -\frac{1}{3} \text{ or } t = 2$$

赵礼显数学

18. (2018·新课标Ⅲ) 函数 $f(x) = \frac{\tan x}{1 + \tan^2 x}$ 的最小正周期为 ()

A. $\frac{\pi}{4}$

B. $\frac{\pi}{2}$

C. π

D. 2π

$$f(x) = \frac{1}{2} \frac{2 \tan x}{1 + \tan^2 x} = \frac{1}{2} \sin 2x \quad T = \frac{2\pi}{2} = \pi$$

赵礼显数学

19. 若 $\alpha \in (0, \pi)$, 且 $\sin \alpha + 2 \cos \alpha = 2$, 则 $\tan \frac{\alpha}{2}$ 等于 ()

A. 3

B. 2

C. $\frac{1}{2}$

D. $\frac{1}{3}$

$$\text{设: } 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 2(1 - \cos \alpha) = 2(1 - 1 + 2 \sin^2 \frac{\alpha}{2})$$

$$\cos \frac{\alpha}{2} = 2 \sin \frac{\alpha}{2} \Rightarrow \tan \frac{\alpha}{2} = \frac{1}{2}$$

$$\text{设: } \frac{2t}{1+t^2} + 2 \cdot \frac{1-t^2}{1+t^2} = 2 \Rightarrow \frac{t+1-t^2}{1+t^2} = 1 \Rightarrow 1+t^2 = t+1-t^2 \Rightarrow 2t^2-t=0 \Rightarrow t=0 \text{ or } t=\frac{1}{2}$$

赵礼显数学

20. (2018·新课标 I) 已知角 α 的顶点为坐标原点, 始边与 x 轴的非负半轴重合, 终边上有两点

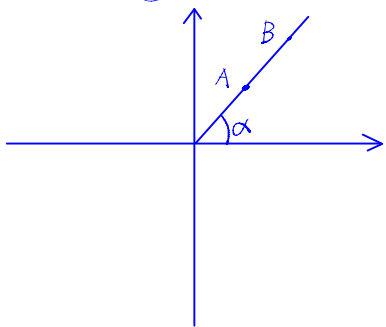
$A(1, a)$, $B(2, b)$, 且 $\cos 2\alpha = \frac{2}{3}$, 则 $|a-b| =$ $\sqrt{\tan^2 \alpha}$

A. $\frac{1}{5}$

B. $\frac{\sqrt{5}}{5}$

C. $\frac{2\sqrt{5}}{5}$

D. 1



$$k = \frac{b-a}{2-1} = \tan \alpha$$

$$\cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{2}{3}$$

$$2 \tan^2 \alpha + 2 = 3 - 3 \tan^2 \alpha$$

$$5 \tan^2 \alpha = 1$$

$$\tan \alpha = \frac{\sqrt{5}}{5}$$

赵礼显数学

21. 已知 $\sin \alpha \sin(\frac{\pi}{3} - \alpha) = 3 \cos \alpha \sin(\alpha + \frac{\pi}{6})$, 则 $\cos(2\alpha + \frac{\pi}{3}) =$ ()

A. $-\frac{\sqrt{3}}{2}$

B. -1

C. $\frac{1}{2}$

D. $\frac{\sqrt{3}}{2}$

$$\tan \alpha = 3 \tan(\alpha + \frac{\pi}{6})$$

$$\tan[\alpha + \frac{\pi}{6} - \frac{\pi}{6}] = 3 \tan(\alpha + \frac{\pi}{6})$$

$$\frac{t - \frac{\sqrt{3}}{3}}{1 + t \cdot \frac{\sqrt{3}}{3}} = 3t$$

$$t - \frac{\sqrt{3}}{3} = 3t + \sqrt{3}t^2$$

$$\sqrt{3}t^2 + 2t + \frac{\sqrt{3}}{3} = 0$$

$$3t^2 + 2\sqrt{3}t + 1 = 0 \Rightarrow (\sqrt{3}t + 1)^2 = 0 \Rightarrow t = -\frac{\sqrt{3}}{3}$$

$$\frac{1 - \tan^2(\alpha + \frac{\pi}{6})}{1 + \tan^2(\alpha + \frac{\pi}{6})}$$

$$\text{另法: } \cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$= \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2}$$

赵礼显数学

题型四：正切恒等式

$$\underline{A+B+C=k\pi}, \quad \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

$$\Rightarrow A+B=k\pi-C \Rightarrow \tan(A+B) = \tan(k\pi-C) = -\tan C$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \tan A + \tan B = -\tan C + \tan A \cdot \tan B \cdot \tan C$$

$$\Rightarrow \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

赵礼显数学

22. (2025·徐州模拟) 已知 $\triangle ABC$ 的内角 A, B, C 满足 $\tan A + \tan B - \tan(A+B) = 2\tan A \tan B$,

则 $\cos^4 C - \sin^4 C = (\quad)$

A. $\frac{4}{5}$

B. $\frac{3}{5}$

C. $-\frac{4}{5}$

D. $-\frac{3}{5}$



$$\tan A + \tan B + \tan C = 2 \tan A \tan B$$

$$\tan C = 2$$

$$(\cos^2 C + \sin^2 C) \cdot (\cos^2 C - \sin^2 C) = \frac{\cos^2 C - \sin^2 C}{\cos^2 C + \sin^2 C} = \frac{1 - \tan^2 C}{1 + \tan^2 C} = \frac{1 - 4}{1 + 4} = -\frac{3}{5}$$

赵礼显数学

23. (2016·江苏) 在锐角三角形 ABC 中, 若 $\sin A = 2\sin B \sin C$, 则 $\tan A \tan B \tan C$ 的最小值是

$$A = \pi - (B+C)$$

$$\sin(B+C) = 2\sin B \sin C$$

$$\sin B \cos C + \cos B \sin C = 2\sin B \sin C \quad \left| \begin{array}{l} \text{同除以} \cos B \cos C \\ \text{得} \end{array} \right.$$

$$\tan B + \tan C = 2 \tan B \tan C \Rightarrow 2 \sqrt{\tan B \tan C} \Rightarrow \tan B \tan C \geq 1$$

$$\tan A \cdot \tan B \cdot \tan C = \tan A + \tan B + \tan C = \tan A + 2 \tan B \tan C \geq 2 \sqrt{2 \tan A \tan B \tan C}$$

$$\Rightarrow \tan A \tan B \tan C \geq 8$$

$$\frac{3}{4} \text{ 且 } \frac{1}{2} \frac{3}{4} \quad \tan A = 2 \tan B \tan C \text{ 取等号}$$

$$\tan B \tan C = 2$$

赵礼显数学

24. 已知 $\triangle ABC$ 为锐角三角形, 且 $\sin A = \sin B \sin C$, 则下列结论中正确的是 ()

A. $\tan B + \tan C = \tan B \tan C$

B. $\tan A \tan B \tan C = \tan A + \tan B + \tan C$

C. $1 < \tan A \leq \frac{4}{3}$

D. $\tan A \tan B \tan C$ 的最小值为 4

$$\sin(B+C) = \sin B \sin C$$

$$\sin B \cos C + \cos B \sin C = \sin B \sin C$$

$$\tan B + \tan C = \tan B \tan C \Rightarrow 2 \sqrt{\tan B \tan C}$$

$$\tan B \tan C \geq 4$$

$$\begin{aligned} D: \tan A \cdot \tan B \cdot \tan C &= \tan A + \tan B + \tan C \\ &= \tan^2 A + \tan^2 B + \tan^2 C \\ &\geq 2 \sqrt{\tan A \tan B \tan C} \\ \therefore \tan A \tan B \tan C &\geq 4 \end{aligned}$$

$$\tan A = -\tan(B+C) = -\frac{\tan B + \tan C}{1 - \tan B \tan C} = \frac{\tan B \tan C - 1 + 1}{\tan B \tan C - 1} = 1 + \frac{1}{\tan B \tan C - 1} \downarrow \in (1, \frac{4}{3}]$$

$$\begin{aligned} D: \tan A + \tan B + \tan C &= 1 + \frac{1}{\tan B \tan C - 1} + \tan B \tan C = 1 + \frac{1}{x} + x + 1 = x + \frac{1}{x} + 2 \geq 3 + \frac{1}{3} + 2 \\ &= 5 + \frac{1}{3} = \frac{16}{3} \end{aligned}$$

赵礼显数学